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Time domain modeling of damping using anelastic displacement fields and fractional calculus

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Abstract

A fractional derivative model of linear viscoelasticity based on the decomposition of the displacement field into an anelastic part and elastic part is developed. The evolution equation for the anelastic part is then a differential equation of fractional order in time. By using a fractional order evolution equation for the anelastic strain the present model becomes very flexible for describing the weak frequency dependence of damping characteristics. To illustrate the modeling capability, the model parameters are fit to available frequency domain data for a high damping polymer. By studying the relaxation modulus and the relaxation spectrum the material parameters of the present viscoelastic model are given physical meaning. The use of this viscoelastic model in structural modeling is discussed and the corresponding finite element equations are outlined, including the treatment of boundary conditions. The anelastic displacement field is mathematically coupled to the total displacement field through a convolution integral with a kernel of Mittag–Leffler function type. Finally a time step algorithm for solving the finite element equations are developed and some numerical examples are presented.

1. Introduction

The present study deals with the modeling of material damping. Energy dissipation is assumed to occur within the continuum element. Material damping, is commonly quantified by a loss factor which is the ratio of the energy lost to the maximum stored energy within the continuum element under steady-state harmonic conditions. For linear materials the loss factor can also be expressed as the ratio between the imaginary and real parts of the complex dynamic modulus. Material damping is often approximated in the engineering community by a complex, frequency-inde-

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pendent dynamic modulus (i.e., by a constant loss factor). When extrapolated to the entire frequency domain, this model represents a non-causal relation between excitation and response (Crandall, 1970). Another model often used due to its mathematical simplicity and convenience is the viscous model. In the viscous model, the stress is taken to be the sum of parts proportional to the strain and the strain rate. In the frequency domain, the imaginary part of the corresponding dynamic modulus is proportional to the frequency. The loss factor is thus proportional to the excitation frequency. Unfortunately, this model is not appropriate to describe the dynamic behavior of a wide class of nearly elastic materials. Instead, it is found experimentally that many engineering materials show loss factors with relatively weak frequency dependence (Kimball and Lovell, 1927).

Time domain viscoelastic models can provide an accurate description of the actual dynamic behavior of materials while guaranteeing causality. At present, these models are not widely used in structural dynamics. The linear constitutive equation of classic viscoelasticity may be formulated in three different forms. The first form contains several integer time derivatives of order one and higher on both stress and strain. Many higher-order time derivatives on both stress and strain are demanded to match the weak frequency dependence of the dynamic properties. This makes the model cumbersome and it leads to higher-order equations of motion when incorporated into structural dynamics. The second form involves convolution or Stieltjes integrals. To match the integer derivative form the kernel should be taken as a sum of exponentially decaying terms. Once more, to match the actual dynamic behavior many parameters are demanded which makes the model cumbersome. The third form uses the concept of internal variables (e.g. taken as augmenting thermodynamic fields (ATF) or anelastic displacement fields (ADF), see Lesieutre and Mingori (1990) and Lesieutre and Bianchini (1995). The constitutive equation is then formulated as a set of coupled equations. The governing differential equation for the internal variable is first-order in time. However, many internal variables must be used to match the weak frequency dependence over a broad frequency range which can make the model cumbersome for use in such circumstances. Dovstam (1995) proposes an Augmented Hooke's law (AHL), which is reported as an extended formulation of classic viscoelasticity, to model three-dimensional damping in frequency domain calculations. This formulation may also require many parameters to describe weak frequency.

Weak frequency dependence of dynamic properties of a viscoelastic material can also be described by using fractional derivative operators in the constitutive relations. Bagley and Torvik (1983a) showed very good agreement using only four parameters when they fit their fractional derivative model of viscoelasticity to experimental data for a polymer. When this model is incorporated directly into a structural dynamics framework it leads to higher order equations of motion (i.e. higher than order two in time). The increased order leads to several mathematical disadvantages when solving the equations of motion (e.g., higher fractional-order initial conditions, see Enelund and Olsson, 1995). To avoid the problem of high-order of equations of motion, a form of the fractional calculus model of viscoelasticity involving a convolution integral with a weakly singular kernel of Mittag–Leffler type can be used, see Enelund and Olsson (1995) and Enelund and Josefson (1997). Little work considers finite element formulations and algorithms for time integration of responses of viscoelastic structures with fractional derivatives constitutive relations. One such work is Padovan (1987). In this work, the viscoelastic behavior is taken in consideration by using a constitutive relation involving fractional derivative operators acting on both stress and strain.

A recent survey article of the research in the application of fractional calculus to solid mechanics is given by Rossikhin and Shitikova (1997).

An alternative viscoelastic formulation reported to be suitable to model the weak frequency dependence of the damping characteristics is the Golla–Hughes–McTavish mini-oscillator model (Golla and Hughes, 1985; McTavish and Hughes, 1993). The mini-oscillator model employs a somewhat different kernel or relaxation modulus compared to classical viscoelasticity (without using fractional calculus operators). Another difference between the present ADF-model and the GHM-model is that the ADF-model is a direct time domain model while the GHM-model is transform-based.

In the present study a three-dimensional fractional derivative model of viscoelasticity is formulated using the concept of internal variables modeled with anelastic displacement fields (ADF) in complete analogy with Lesieutre and Bianchini (1995). The evolution equations for the internal fields then contain general order derivative operators instead of first-order integer derivative operators as in the initial ADF-model.

To illustrate the ability of the model to describe weak frequency dependence model parameters are fitted to complex modulus data for a high damping polymer. Time domain expressions for the corresponding stress relaxation modulus and the relaxation spectrum are presented. The use of this fractional-order anelastic displacement field model in structural dynamics modeling is discussed and the corresponding finite element equations are formulated. Finally, a time step algorithm for solving the finite element equations is outlined and some numerical examples are given.

2. Linear viscoelasticity with anelastic displacement fields

We will now develop a time domain linear viscoelasticity model based on the decomposition of the displacement field into an elastic and an anelastic part to model dissipative behavior. Isothermal conditions are assumed throughout the present study. In general all field variables are functions of time and space. However, when discussing constitutive relations the space dependence will be suppressed. Cartesian components of tensors are denoted by subscript indices, while a subscript index within parentheses denotes a specific variable or set of variables.

The total displacement is taken to be the sum of elastic and anelastic displacements (Lesieutre and Bianchini, 1995)

$$\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{u}^{\mathrm{E}}(\boldsymbol{x},t) + \boldsymbol{u}^{\mathrm{A}}(\boldsymbol{x},t) \tag{1}$$

where $\boldsymbol{u}^{\mathrm{E}}$ is the elastic displacement field and $\boldsymbol{u}^{\mathrm{A}}$ is the anelastic displacement field which may be expressed as a sum of N individual anelastic fields that correspond to different relaxation processes:

$$\boldsymbol{u}^{\mathrm{A}}(\boldsymbol{x},t) = \sum_{n=1}^{N} \boldsymbol{u}^{\mathrm{A}}_{(n)}(\boldsymbol{x},t) \tag{2}$$

We assume here equal spatial variation of the two displacement fields. The rotations and the strains are assumed to be small (i.e., $\varepsilon_{ij}^2 \ll 1$) and the (infinitesimal) strain tensor is defined by (in tensor notation)

$$\begin{aligned}\varepsilon_{ij}(t) &= 1/2(u_{i,j}(t)+u_{j,i}(t)) = 1/2(u^{\mathrm{E}}_{i,j}(t)+u^{\mathrm{E}}_{j,i}(t))+1/2(u^{\mathrm{A}}_{i,j}(t)+u^{\mathrm{A}}_{j,i}(t)) \\ &= \varepsilon^{\mathrm{E}}_{ij}(t)+\varepsilon^{\mathrm{A}}_{ij}(t) \end{aligned} \tag{3}$$

where differentiation is with respect to the space-coordinate and $\varepsilon^{\mathrm{E}}_{ij}$ is the elastic strain tensor and $\varepsilon^{\mathrm{A}}_{ij}$ is the anelastic or creep strain tensor which also may be expressed as a sum of N individual anelastic strains:

$$\varepsilon^{\mathrm{A}}_{ij}(t) = \sum_{n=1}^{N} \varepsilon^{\mathrm{A}}_{ij(n)}(t) \tag{4}$$

For convenience, strains and stresses are also expressed using compressed matrix notation (e.g., $\boldsymbol{\varepsilon} = [\varepsilon_1 \quad \varepsilon_2 \quad \varepsilon_3 \quad \varepsilon_4 \quad \varepsilon_5 \quad \varepsilon_6]^{\mathrm{T}}$ and $\boldsymbol{\sigma} = [\sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \sigma_4 \quad \sigma_5 \quad \sigma_6]^{\mathrm{T}}$) following the ordering conventions

$$\varepsilon_1 = \varepsilon_{11}, \quad \varepsilon_2 = \varepsilon_{22}, \quad \varepsilon_3 = \varepsilon_{33}, \quad \varepsilon_4 = 2\varepsilon_{23}, \quad \varepsilon_5 = 2\varepsilon_{31}, \quad \varepsilon_6 = 2\varepsilon_{12}$$

and

$$\sigma_1 = \sigma_{11}, \quad \sigma_2 = \sigma_{22}, \quad \sigma_3 = \sigma_{33}, \quad \sigma_4 = \sigma_{23}, \quad \sigma_5 = \sigma_{31}, \quad \sigma_6 = \varepsilon_{12}$$

The set of coupled equations forming the constitutive relation is determined from the following quadratic Helmholtz free energy function (Lesieutre and Bianchini, 1995)

$$f = \frac{1}{2}\left(E_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - 2E_{ijkl}\varepsilon_{ij}\sum_{n=1}^{N}\varepsilon^{\mathrm{A}}_{kl(n)} + \sum_{n=1}^{N} E^{\mathrm{A}}_{ijkl(n)}\varepsilon^{\mathrm{A}}_{ij(n)}\varepsilon^{\mathrm{A}}_{kl(n)} \right) \tag{5a}$$

or in compressed matrix notation

$$f = \frac{1}{2}\begin{bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\varepsilon}^{\mathrm{A}}_1 \\ \cdots \\ \boldsymbol{\varepsilon}^{\mathrm{A}}_n \\ \cdots \\ \boldsymbol{\varepsilon}^{\mathrm{A}}_N \end{bmatrix} \begin{bmatrix} \boldsymbol{E} & -\boldsymbol{E} & \vdots & -\boldsymbol{E} & \vdots & -\boldsymbol{E} \\ -\boldsymbol{E} & \boldsymbol{E}^{\mathrm{A}}_1 & \vdots & \mathbf{0} & \vdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\boldsymbol{E} & \mathbf{0} & \vdots & \boldsymbol{E}^{\mathrm{A}}_n & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\boldsymbol{E} & \mathbf{0} & \vdots & \mathbf{0} & \vdots & \boldsymbol{E}^{\mathrm{A}}_N \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\varepsilon}^{\mathrm{A}}_1 \\ \cdots \\ \boldsymbol{\varepsilon}^{\mathrm{A}}_n \\ \cdots \\ \boldsymbol{\varepsilon}^{\mathrm{A}}_N \end{bmatrix} \tag{5b}$$

where E_{ijkl} denotes the fourth-order tensor of instantaneous or unrelaxed material parameters and $E^{\mathrm{A}}_{ijkl(n)}$ denotes the fourth-order tensor of anelastic material parameters corresponding to the n-th anelastic strain, while $\boldsymbol{E}$ is the matrix representation of E_{ijkl} and $\boldsymbol{E}^{\mathrm{A}}_n$ are the matrix representations of $E^{\mathrm{A}}_{ijkl(n)}$. The two-coupled set of material constitutive equations can now be found from

$$\sigma_{ij} = \frac{\partial f}{\partial \varepsilon_{ij}} \quad \text{or } \boldsymbol{\sigma} = \frac{\partial f}{\partial \boldsymbol{\varepsilon}} \tag{6a}$$

and

$$\sigma^{\mathrm{A}}_{ij(n)} = -\frac{\partial f}{\partial \varepsilon^{\mathrm{A}}_{ij(n)}} \quad \text{or } \boldsymbol{\sigma}^{\mathrm{A}}_{(n)} = -\frac{\partial f}{\partial \boldsymbol{\varepsilon}^{\mathrm{A}}_{(n)}} \tag{6b}$$

where $\sigma^{\mathrm{A}}_{ij(n)}$ are the anelastic stress quantities conjugate to $\varepsilon^{\mathrm{A}}_{ij(n)}$. Application of eqn (6) leads to the two set of constitutive relations:

$$\sigma_{ij} = E_{ijkl}\left(\varepsilon_{kl} - \sum_{n=1}^{N} \varepsilon^{\mathrm{A}}_{kl(n)}\right) \quad \text{or } \boldsymbol{\sigma} = \boldsymbol{E}\left(\boldsymbol{\varepsilon} - \sum_{n=1}^{N} \boldsymbol{\varepsilon}^{\mathrm{A}}_{(n)}\right) \tag{7a}$$

$$\sigma^{\mathrm{A}}_{ij(n)} = E_{ijkl}\varepsilon_{kl} - E^{\mathrm{A}}_{ijkl(n)}\varepsilon^{\mathrm{A}}_{kl(n)} \quad \text{or } \boldsymbol{\sigma}^{\mathrm{A}}_{(n)} = \boldsymbol{E}\boldsymbol{\varepsilon} - \boldsymbol{E}^{\mathrm{A}}_{(n)}\boldsymbol{\varepsilon}^{\mathrm{A}}_{(n)}, \quad n = 1, \ldots, N \tag{7b}$$

We now limit ourselves to isotropic materials. In that case it is convenient to introduce the deviatoric parts s_{ij} of the stress tensor σ_{ij}, and e_{ij} of the strain tensor ε_{ij}

$$s_{ij} = \sigma_{ij} - \tfrac{1}{3}\delta_{ij}\sigma_{kk}, \quad s_{ii} = 0 \tag{8a}$$

$$e_{ij} = \varepsilon_{ij} - \tfrac{1}{3}\delta_{ij}\varepsilon_{kk}, \quad e_{ii} = 0 \tag{8b}$$

where δ_{ij} is the Kronecker delta symbol, $\frac{1}{3}\delta_{ij}\sigma_{kk}$ and $\frac{1}{3}\delta_{ij}\varepsilon_{kk}$ are the hydrostatic parts of the stress tensor and the strain tensor, respectively. Using this decomposition and assuming isotropic anelastic behavior, we write the constitutive equation for the stress as

$$s_{ij}(t) = 2G\left(e_{ij}(t) - \sum_{n=1}^{N} e^{\mathrm{A}}_{ij(n)}(t)\right) \tag{9a}$$

$$\sigma_{kk}(t) = 3K\left(\varepsilon_{kk}(t) - \sum_{n=1}^{N} \varepsilon^{\mathrm{A}}_{kk(n)}(t)\right) \tag{9b}$$

where G is the instantaneous shear modulus and K is the instantaneous bulk modulus. The constitutive equations for the anelastic stress quantities can also be written as uncoupled equations for the deviatoric and hydrostatic parts

$$s^{\mathrm{A}}_{ij(n)}(t) = 2Ge_{ij}(t) - 2G^{\mathrm{A}}_{(n)}e^{\mathrm{A}}_{ij(n)}(t), \quad n = 1, \ldots, N \tag{10a}$$

$$\sigma^{\mathrm{A}}_{kk(n)}(t) = 3K\varepsilon_{kk}(t) - 3K^{\mathrm{A}}_{(n)}\varepsilon^{\mathrm{A}}_{kk(n)}(t), \quad n = 1, \ldots, N \tag{10b}$$

where $G^{\mathrm{A}}_{(n)}$ and $K^{\mathrm{A}}_{(n)}$ are the anelastic shear and bulk moduli corresponding to the n-th anelastic strain, respectively. The constitutive equation for the stresses can be expressed in matrix notation as

$$\boldsymbol{\sigma}(t) = K\boldsymbol{M}_{\mathrm{K}}\left(\boldsymbol{\varepsilon}(t) - \sum_{n=1}^{N} \boldsymbol{\varepsilon}^{\mathrm{A}}_{(n)}(t)\right) + G\boldsymbol{M}_{\mathrm{G}}\left(\boldsymbol{\varepsilon}(t) - \sum_{n=1}^{N} \boldsymbol{\varepsilon}^{\mathrm{A}}_{(n)}(t)\right) \tag{11}$$

and

$$\boldsymbol{\sigma}^{\mathrm{A}}_{n}(t) = \boldsymbol{M}_{\mathrm{K}}(K\boldsymbol{\varepsilon}(t) - K^{\mathrm{A}}_{(n)}\boldsymbol{\varepsilon}^{\mathrm{A}}_{(n)}(t)) + \boldsymbol{M}_{\mathrm{G}}(G\boldsymbol{\varepsilon}(t) - G^{\mathrm{A}}_{(n)}\boldsymbol{\varepsilon}^{\mathrm{A}}_{(n)}(t)) \tag{12}$$

where

$$\boldsymbol{M}_{\mathrm{K}} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{M}_{\mathrm{G}} = \begin{bmatrix} 4/3 & -2/3 & -2/3 & 0 & 0 & 0 \\ -2/3 & 4/3 & -2/3 & 0 & 0 & 0 \\ -2/3 & -2/3 & 4/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

2.1. Relaxation equation

The evolution or relaxation equations for the anelastic strains are in classical viscoelasticity taken as first-order differential equations in time (cf. Lesieutre and Mingori, 1990; Enelund and Olsson, 1995). By using fractional-order derivative operators in the evolution equations for the anelastic strains, the viscoelastic model is potentially very useful for describing observed material behavior accurately with few material parameters (cf. Bagley and Torvik, 1983a). The key effect is that whole spectrum of dissipative mechanisms may be included in a single anelastic strain. The fractional-order evolution equations for the anelastic strains are then written as (cf Enelund and Olsson, 1995)

$$\mathrm{D}^{\alpha}{}_{\mathrm{G}(n)} e^{\mathrm{A}}_{ij(n)}(t) = -\frac{1}{b^{\alpha_{\mathrm{G}(n)}}_{\mathrm{G}(n)}}\left(e^{\mathrm{A}}_{ij(n)}(t) - \frac{\Delta G_{(n)}}{G} e_{ij}(t)\right), \quad 0 < \alpha_{\mathrm{G}(n)} \leqslant 1 \quad n = 1, \ldots, N \tag{13a}$$

$$\mathrm{D}^{\alpha}{}_{\mathrm{K}(n)} \varepsilon^{\mathrm{A}}_{kk(n)}(t) = -\frac{1}{b^{\alpha_{\mathrm{K}(n)}}_{\mathrm{K}(n)}}\left(\varepsilon^{\mathrm{A}}_{kk}(t) - \frac{\Delta K_{(n)}}{K} \varepsilon_{kk}(t)\right), \quad 0 < \alpha_{\mathrm{K}(n)} \leqslant 1 \quad n = 1, \ldots, N \tag{13b}$$

with

$$\Delta G_{(n)} = G^2 / G^{\mathrm{A}}_{(n)} \quad \text{and} \quad \Delta K_{(n)} = K^2 / K^{\mathrm{A}}_{(n)}$$

where $(\Delta G_{(n)}/G) e_{ij}$ is what makes $s_{ij(n)}$ zero (i.e., the strain-dependent equilibrium value of $e^{\mathrm{A}}_{ij(n)}$) and $(\Delta K_{(n)}/K)\varepsilon_{kk}$ is what makes $\sigma_{kk(n)}$ zero (i.e., the strain-dependent equilibrium value of $\varepsilon^{\mathrm{A}}_{kk(n)}$). Moreover, $b_{\mathrm{G}(n)}$ and $\alpha_{\mathrm{G}(n)}$ are the relaxation constant and the fractional-order of differentiation in pure shear, while $b_{\mathrm{K}(n)}$ and $\alpha_{\mathrm{K}(n)}$ are the relaxation constant and fractional-order of differentiation in hydrostatic deformation and D^{α} is the generalized derivative operator of order α. A suitable definition of differentiation of fractional-order for the application to viscoelasticity is (Oldham and Spanier, 1974)

$$\mathrm{D}^{\alpha} x(t) \equiv \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^t \frac{x(\tau)}{(t-\tau)^{\alpha}} \mathrm{d}\tau\right], \quad 0 < \alpha < 1 \tag{14}$$

where Γ is the gamma function. From the definition of fraction-order differentiation above we note that the fractional derivative operator is not a temporally local operator as the ordinary integer derivative operator is, i.e., when calculating the integer-order derivative of function x it is sufficient to know the function in an arbitrarily small interval close to t, but when calculating the

fractional-order derivative of function x it is necessary to know the function in the whole interval $(0, t)$.

Because eqns (13a) and (13b) are differential equations initial conditions are required. The formal initial conditions are found to be (Enelund and Olsson, 1995)

$$\mathrm{D}^{-(1-\alpha_{\mathrm{G}(n)})} e^{\mathrm{A}}_{ij(n)}(0^+) = 0, \quad n = 1, \ldots, N \tag{15a}$$

$$\mathrm{D}^{-(1-\alpha_{\mathrm{K}(n)})} \varepsilon^{\mathrm{A}}_{kk(n)}(0^+) = 0, \quad n = 1, \ldots, N \tag{15b}$$

These initial conditions are (for $\alpha_{\mathrm{G}(n)}, \alpha_{\mathrm{K}(n)} \in (0, 1)$) integral conditions and allow for unbounded strain histories. However, from a physical point of view, unbounded strains must be disqualified. The initial conditions to eqns (13a) and (13b) can then be taken as

$$e^{\mathrm{A}}_{ij(n)}(0^+) = 0, \quad n = 1, \ldots, N \tag{16a}$$

$$\varepsilon^{\mathrm{A}}_{kk(n)}(0^+) = 0, \quad n = 1, \ldots, N \tag{16b}$$

which are consistent with an instantaneous response at time $t = 0^+$ following Hooke's elastic law as seen from eqns (9a) and (9b) by introducing the initial conditions in eqns (16a) and (16b).

Consider a single anelastic displacement field ($N = 1$). This field may itself model a whole spectrum of dissipative mechanisms. The anelastic strains can be eliminated from eqns (9a) and (10a) by using the evolution equations in eqns (13a) and (13b). The deviatoric part is first eliminated by applying the operator $\mathrm{D}^{\alpha_{\mathrm{G}}}$ to eqn (9a) and then using eqn (9a) in combination with eqn (13a). The hydrostatic part is eliminated in the same manner. The results are:

$$s_{ij}(t) + b^{\alpha_{\mathrm{G}}}_{\mathrm{G}} \mathrm{D}^{\alpha_{\mathrm{G}}} s_{ij}(t) = 2G_\infty e_{ij}(t) + 2G b^{\alpha_{\mathrm{G}}}_{\mathrm{G}} \mathrm{D}^{\alpha_{\mathrm{G}}} e_{ij}(t) \tag{17a}$$

$$\sigma_{kk}(t) + b^{\alpha_{\mathrm{K}}}_{\mathrm{K}} \mathrm{D}^{\alpha_{\mathrm{K}}} \sigma_{kk}(t) = 3K_\infty \varepsilon_{kk}(t) + 3K b^{\alpha_{\mathrm{K}}}_{\mathrm{K}} \mathrm{D}^{\alpha_{\mathrm{K}}} \varepsilon_{kk}(t) \tag{17b}$$

where $G_\infty = G - \Delta G$ is the long-time or relaxed shear modulus and $K_\infty = K - \Delta K$ is the long time or relaxed bulk modulus. Equations (17a) and (17b) are commonly referred to as the fractional calculus model of viscoelasticity (here generalized to three-dimensional isotropic stress states). We may now conclude that in the case of a single anelastic strain eqns (9a) and (10a) together with eqns (13a) and (13b) are sufficient to describe the same viscoelastic behavior as eqns (17a) and (17b).

In general relaxation times (b_{K} and b_{G}) and the fractional derivative exponents (α_{K} and α_{G}) need not to be equal. However, for a polycrystalline material with randomly oriented crystal directions the order of fractional differentiations and the relaxation constants in deviatoric and hydrostatic relaxation can be assumed to be equal (i.e., $\alpha_{\mathrm{G}} = \alpha_{\mathrm{K}} = \alpha$ and $b_{\mathrm{G}} = b_{\mathrm{K}} = b$). The relaxation equations [eqns (13b) and (13a)] can then be written as the single matrix equation:

$$\mathrm{D}^{\alpha} \boldsymbol{\varepsilon}^{\mathrm{A}}(t) = -\frac{1}{b^{\alpha}} \left(\boldsymbol{\varepsilon}^{\mathrm{A}}(t) - \left(\frac{\Delta K}{3K} \boldsymbol{M}_{\mathrm{K}} + \frac{\Delta G}{2G} \boldsymbol{M}_{\mathrm{C}} \right) \boldsymbol{\varepsilon}(t) \right) \tag{18}$$

where

$$\boldsymbol{M}_{\mathrm{C}} = \begin{bmatrix} 4/3 & -2/3 & -2/3 & 0 & 0 & 0 \\ -2/3 & 4/3 & -2/3 & 0 & 0 & 0 \\ -2/3 & -2/3 & 4/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

with the initial condition

$$\boldsymbol{\varepsilon}^{\mathrm{A}}(0^{+}) = \mathbf{0} \tag{19}$$

Note that $\Delta K = \Delta G = 0$ yield $e_{ij}^{\mathrm{A}} = \varepsilon_{kk}^{\mathrm{A}} = 0$ or $\boldsymbol{\varepsilon}^{\mathrm{A}} = \mathbf{0}$, since 'eigenfunctional' solutions [i.e., the solution to $\mathrm{D}^{\alpha}\varepsilon^{\mathrm{A}}(t) + (1/b)\varepsilon^{\mathrm{A}}(t) = 0$ which, in principle, is the memory kernel in, e.g., eqn (20a)] are disqualified by the initial conditions [e.g., eqns (15a)].

2.2. Convolution integral formulation

The anelastic strains can be interpreted as convolution integrals with the strain and a singular kernel of Mittag–Leffler function type. By applying a Laplace transformation and a subsequent inversion to eqns (13a) and (13b) with the initial conditions in eqns (15a) and (15b), we obtain (see Enelund and Olsson, 1995)

$$e_{ij(n)}^{\mathrm{A}}(t) = \frac{\Delta G_{(n)}}{G} \int_0^t f_{1(n)}(t-\hat{t}) e_{ij}\, \mathrm{d}\hat{t} \tag{20a}$$

with

$$f_{1(n)} = -\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{E}_{\alpha_{\mathrm{G}(n)}}[-(t/b_{\mathrm{G}(n)})^{\alpha_{\mathrm{G}(n)}}]), \quad t > 0$$

$$\varepsilon_{kk(n)}^{\mathrm{A}}(t) = \frac{\Delta K_{(n)}}{K} \int_0^t f_{2(n)}(t-\hat{t}) \varepsilon_{kk}\, \mathrm{d}\hat{t} \tag{20b}$$

with

$$f_{2(n)} = -\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{E}_{\alpha_{\mathrm{K}(n)}}[-(t/b_{\mathrm{K}(n)})^{\alpha_{\mathrm{K}(n)}}]), \quad t > 0$$

where $f_{1(n)}(t)$ and $f_{2(n)}$ are the memory kernels in pure shear and isotropic compression, respectively, while E_{α} is the α-order Mittag–Leffler function which is defined as (Bateman, 1955)

$$\mathrm{E}_{\alpha}(u) = \sum_{k=0}^{\infty} \frac{u^k}{\Gamma(1+\alpha k)} \tag{21}$$

Consider the case of $\alpha_{\mathrm{G}(n)} = \alpha_{\mathrm{K}(n)} = 1$, then the memory kernels $f_{1(n)}(t)$ and $f_{2(n)}(t)$ in eqns (13a) and (13b) become

$$f_{1(n)}(t) = \frac{1}{b_{\mathrm{G}(n)}} \frac{\Delta G_{(n)}}{G} \mathrm{e}^{-t/b_{\mathrm{G}(n)}}, \quad t > 0 \tag{22a}$$

$$f_{2(n)}(t) = \frac{1}{b_{\mathrm{K}(n)}} \frac{\Delta K_{(n)}}{K} \mathrm{e}^{-t/b_{\mathrm{K}(n)}}, \quad t > 0 \tag{22b}$$

which are the memory kernels corresponding to the classical Standard Linear Viscoelastic Solid.

Kernels functions of Mittag–Leffler function type were first introduced into viscoelasticity of solids by Rabotnov (1980). The connection between the convolution integral description of viscoelastic with kernels of Mittag–Leffler function type and fractional calculus model of viscoelasticity in operator form [cf. eqns (17a) and (17b)] was established by Koeller (1984). Convolution kernels of Mittag–Leffler type in dynamic analysis are also discussed in Tseitlin and Kusainov (1990).

3. Viscoelastic functions

We now consider uniaxial constitutive behavior described by the following set of equations (in analogy with the isotropic three-dimensional description above)

$$\sigma(t) = E\left(\varepsilon(t) - \sum_{n=1}^{N} \varepsilon^{\mathrm{A}}_{(n)}(t)\right) \tag{23}$$

and

$$\mathrm{D}^{\alpha_{(n)}} \varepsilon^{\mathrm{A}}_{(n)}(t) = -\frac{1}{b^{\alpha_{(n)}}_{(n)}} \left(\varepsilon^{\mathrm{A}}_{(n)}(t) - \frac{\Delta E_{(n)}}{E} \varepsilon(t)\right), \quad 0 < \alpha \leqslant 1 \quad n = 1, \ldots, N \tag{24}$$

together with the initial condition

$$\varepsilon_{(n)}(0^+) = 0 \tag{25}$$

where E is the instantaneous modulus and $\Delta E_{(n)}$ is the relaxation strength corresponding to the n-th anelastic strain related to the long time modulus through $E_\infty = E - \Sigma_{n=1}^{N} \Delta E_{(n)}$.

3.1. On the dynamic modulus

The (complex) dynamic modulus $E^*(\omega)$ is defined by

$$\sigma(\omega) = E^*(\omega)\varepsilon(\omega) \tag{26}$$

where $\sigma(\omega)$ and $\varepsilon(\omega)$ are the Fourier transforms of stress and strain. By applying a Fourier transformation to eqns (23) and (24) and eliminating the anelastic strains, we obtain the corresponding dynamic modulus as

$$E^*(\omega) = \left(E - \sum_{n=1}^{N} \frac{\Delta E_{(n)}}{1 + (b_{(n)}\mathrm{i}\omega)^{\alpha_{(n)}}}\right) \tag{27}$$

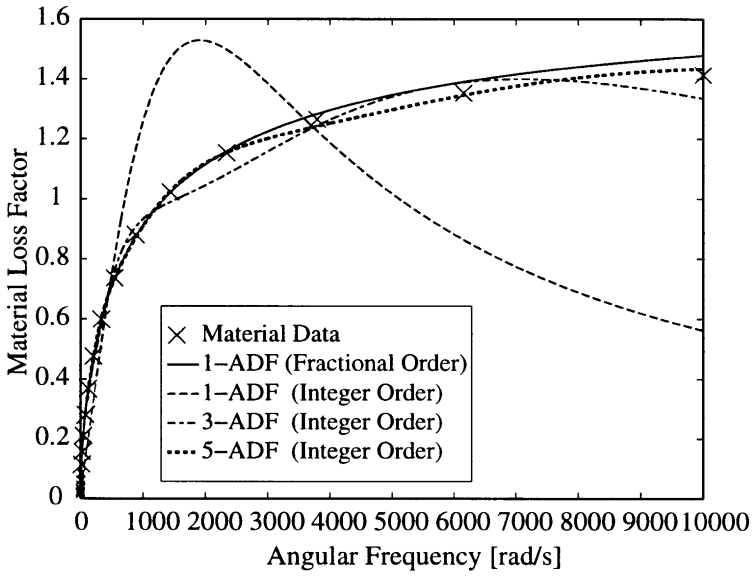

Fig. 1. Measured and predicted material loss factor for a high damping polymer. X-marks are from tabulated material data, the continuous line is curved-fitted ADF-model with a single fractional order anelastic displacement field, the dashed line, the dash–dotted line and the dotted line are curve fitted ADF-models with one, three, and five integer-order anelastic displacement fields.

with the understanding that $(i\omega)^\alpha$ should be interpreted as $\exp(i\alpha\pi/2)(\omega - i0^+)^\alpha$ where (see Gel'fand and Shilov, 1964)

$$(\omega - i0^+)^\alpha = \begin{cases} e^{-i\alpha\pi}|\omega|^\alpha & \omega < 0 \\ \omega^\alpha & \omega > 0 \end{cases} \tag{28}$$

By using a fractional-order relaxation equation the often observed weak frequency dependence of the dynamic modulus can be modeled accurately over several frequency decades with a single anelastic field, while the use of integer-order relaxation functions will require several anelastic strains to model the same behavior. To illustrate this material loss factors of the present model (using one, three, and five, integer-order anelastic strains and one fractional-order anelastic strain) are fit to tabulated loss factor data for the high damping polymer ISD112 (see Soovere and Drake, 1985; Lesieutre and Bianchini, 1995). The experimental data are taken at room temperature and over a frequency range from $\omega = 1$ to $\omega = 10 \times 10^3$ rad/s. Figure 1 shows the material loss factor [i.e., the ratio between the imaginary part and real part of the dynamic modulus $E^*(\omega)$] for ISD112 and the predicted model results using one, three, and five, integer-order anelastic fields (i.e., the evolution equations for the anelastic strains are first-order differential equations) and one fractional-order anelastic displacement (i.e., the evolution equations for the anelastic strain are fractional-order differential equations). The model parameters are chosen so that difference between the material data and model data are minimized in a least square sense. The corresponding material

Table 1
Curve-fitted material parameters for ISD112 material

One fractional-order ADF	$\Delta E = 1200$ MPa	$b = 24.5 \times 10^{-9}$ s $\alpha = 0.675$
$E = 1200.4$ MPa, $E_\infty = 0.431$ MPa		
One integer-order ADF	$\Delta E = 4.94$ MPa	$b = 156 \times 10^{-6}$ s
$E = 5.42$ MPa, $E_\infty = 0.482$ MPa		
Three integer-order ADF	$\Delta E_{(1)} = 0.147$ MPa	$b_{(1)} = 11.3 \times 10^{-3}$ s
$E = 12.1$ MPa, $E_\infty = 0.444$ MPa	$\Delta E_{(2)} = 0.753$ MPa	$b_{(2)} = 0.763 \times 10^{-3}$ s
	$\Delta E_{(3)} = 10.7$ MPa	$b_{(3)} = 42.1 \times 10^{-6}$ s
Five integer-order ADF	$\Delta E_{(1)} = 0.025$ MPa	$b_{(1)} = 0.116$ s
$E = 16.5$ MPa, $E_\infty = 0.435$ MPa	$\Delta E_{(2)} = 0.089$ MPa	$b_{(2)} = 12.4 \times 10^{-3}$ s
	$\Delta E_{(3)} = 0.290$ MPa	$b_{(3)} = 1.91 \times 10^{-3}$ s
	$\Delta E_{(4)} = 1.03$ MPa	$b_{(4)} = 0.317 \times 10^{-3}$ s
	$\Delta E_{(5)} = 14.7$ MPa	$b_{(5)} = 25.7 \times 10^{-6}$ s

parameters are given in Table 1 where $E_\infty = E - \Sigma_{n=1}^{N} \Delta E_{(n)}$ is the relaxed (or long-time, low-frequency) modulus. As seen in Fig. 1, several anelastic fields and comparatively large number of parameters must be used to match measured data, while a single anelastic strain governed by a fractional-order equation seems to be sufficient to model measured data in the specific frequency range.

Since the models are fitted in a specific frequency range, model accuracy outside the frequency range is not guaranteed. We may note from Table 1 the considerable difference in the instantaneous modulus between the integer-order models and the fractional-order model. In fact, the instantaneous modulus and the long-time modulus of the fractional-order ADF model in Table 1 are the same as the experimentally found values for ISD112 (see Soovere and Drake, 1985). This indicates that this fractional-order ADF model can predict instantaneous transient response of the material more accurately.

3.2. On the stress relaxation modulus

The uniaxial stress relaxation modulus $G(t)$ is defined as the stress response to a unit step strain applied at time $t = 0$. The (uniaxial) stress relaxation modulus is here derived by a Laplace transform technique. The Laplace transform of the fractional-order derivative is (see Oldham and Spanier, 1974)

$$\mathscr{L}[\mathrm{D}^\alpha x(t)](s) = s^\alpha \mathscr{L}[x(t)](s) - \sum_{k=0}^{n-1} s^k \mathrm{D}^{\alpha-1-k} x(0^+) \tag{29}$$

where n is an integer such that $n-1 < \alpha \leqslant n$. For simplicity we consider a single anelastic strain. By applying a Laplace transformation to eqn (24), we obtain the Laplace domain anelastic strain response to an imposed unit strain as

$$\hat{\varepsilon}_{\mathrm{u}}^{\mathrm{A}}(s) = \frac{\Delta E}{E} \frac{1}{s(1+(bs)^\alpha)} \tag{30}$$

and the time domain unit response can be found as the inverse Laplace transform

$$\varepsilon_{\mathrm{u}}^{\mathrm{A}}(t) = \frac{\Delta E}{E}\mathscr{L}^{-1}\left[\frac{1}{s(1+(bs)^{\alpha})}\right](t), \quad t>0 \tag{31}$$

To obtain the inverse above we use the same approach as used by Enelund and Olsson (1995) to obtain the memory kernel. Take $c>0$ so that $|(bc)^{-\alpha}|<1$. Along the vertical line $c-\mathrm{i}\infty$ to $c+\mathrm{i}\infty$ in the s-plane we can write

$$\frac{1}{s(1+(bs)^{\alpha})} = b\sum_{k=0}^{\infty}(-1)^k(bs)^{-[\alpha(k+1)+1]} \tag{32}$$

which converges uniformly along the line in consideration. The unit response $\varepsilon_{\mathrm{u}}^{\mathrm{A}}(t)$ may now be written as the line integral:

$$\begin{aligned}\varepsilon_{\mathrm{u}}^{\mathrm{A}}(t) &= \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\varepsilon_{\mathrm{u}}^{\mathrm{A}}(s)\mathrm{e}^{st}\,\mathrm{d}s = \sum_{k=0}^{\infty}(-1)^k\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{\Delta E}{E}b(bs)^{-[\alpha(k+1)+1]}\mathrm{e}^{st}\,\mathrm{d}s\\ &= \frac{\Delta E}{E}\sum_{k=0}^{\infty}(-1)^k\mathscr{L}^{-1}[b(bs)^{-[\alpha(k+1)+1]}](t), \quad t>0\end{aligned} \tag{33}$$

The inverse of each term above is found in a standard table of inverse Laplace transforms (e.g., Oberhettinger and Badii, 1973) and we obtain

$$\varepsilon_{\mathrm{u}}^{\mathrm{A}}(t) = \frac{\Delta E}{E}\sum_{k=0}^{\infty}(-1)^k\frac{(t/b)^{\alpha(k+1)}}{\Gamma(\alpha(k+1)+1)} = \frac{\Delta E}{E}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(t/b)^{\alpha k}}{\Gamma(\alpha k+1)}, \quad t>0 \tag{34}$$

Insertion of $\varepsilon_{\mathrm{u}}^{\mathrm{A}}(t)$ into eqn (23) gives the stress response to a unit strain or the stress relaxation function as

$$G(t) = E-\Delta E\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(t/b)^{\alpha k}}{\Gamma(\alpha k+1)}, \quad t>0 \tag{35}$$

or, in Mittag–Leffler notation

$$G(t) = E-\Delta E(1-E_{\alpha}[-(t/b)^{\alpha}]), \quad t>0 \tag{36}$$

which can be generalized to N anelastic fields as

$$G(t) = E-\sum_{n=1}^{N}\Delta E_{(n)}(1-E_{\alpha_{(n)}}[-(t/b_{(n)})^{\alpha_{(n)}}]), \quad t>0 \tag{37}$$

In the special case of $\alpha_{(n)}=1$, we have

$$G(t) = E-\sum_{n=1}^{N}\Delta E_{(n)}(1-\mathrm{e}^{-t/b_{(n)}}) \tag{38}$$

The series of exponentials in eqn (38) is known as the Prony Series. Using Stirling's formula for asymptotic expansion of the gamma function for large arguments the asymptotic behavior of the k-th term (S_k) in the sum in eqn (35) as $k\to\infty$ is found as

$$S_k = (-1)^{k+1}\frac{(t/b)^{\alpha k+1}}{\Gamma(\alpha k+1)} \sim \frac{(-1)^{k+1}(t/b)^{\alpha k}}{\sqrt{(2\pi)}\,\mathrm{e}^{-(\alpha k+1)}(\alpha k+1)^{\alpha k+1/2}} \tag{39}$$

For a given time the ratio test gives

$$\left|\frac{S_{k+1}}{S_k}\right| \sim \left(\frac{t/b}{\alpha k+\alpha+1}\right)^{\alpha}, \quad \text{as } k \to \infty \tag{40}$$

The series is convergent but the convergence is poor and we need to add

$$k > \frac{t/b-1-\alpha}{\alpha} \tag{41}$$

before the terms begin to fall in magnitude. Due to the rather poor convergence of the series the expression for the relaxation modulus eqn (35) is not of practical use for increasing times and there is a need for an asymptotic expansion of $G(t)$ for large times. To obtain such an expansion, we utilize the same mode of reasoning used by Enelund and Olsson (1995). Taking the Laplace transform of $G(t)$ according to eqn (35) and expanding it for small $s > 0$ yields

$$G(s) \sim \left(\frac{E}{s} - \Delta E b \sum_{k=0}^{\infty}(-1)^k (bs)^{\alpha k-1}\right), \quad s \to 0 \tag{42}$$

A formal term-wise inversion yields the time domain asymptotic expansion of $G(t)$ as $t \to \infty$ as

$$G(t) \sim E - \Delta E\left(1+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(t/b)^{-\alpha k}}{\Gamma(-\alpha k+1)}\right), \quad t \to \infty \tag{43}$$

where any term with $\alpha k-1$ equal to an integer drops out of the sum. (They represent terms with a concentrated support at $t = 0$.) The definition of the gamma function is extended to negative non-integer arguments by analytical continuation. The short time and long time values of the stress relaxation modulus can now be found from eqns (35) and (43) as

$$\lim_{t\to 0^+} G(t) = E \quad \text{and} \quad \lim_{t\to\infty} G(t) = E_\infty \tag{44}$$

Figure 2 shows the stress relaxation modulus according to eqn (35) for different values of the fractional derivative exponent α and for $E/E_\infty = 2$. The asymptotic expression in eqn (43) is used then displaying the stress relaxation modulus for large times. By employing a fractional-order derivative exponent in the relaxation equation the present viscoelastic model becomes capable of describing stress relaxation over many time decades. Further, the viscoelastic parameters can be found from time domain relaxation tests.

3.3. On the relaxation spectrum

The relaxation spectrum is an alternative way of representing the mechanical properties of a viscoelastic material. The relaxation spectrum, $H(\tau)$ shows the distribution of relaxation times and is usually defined in terms of the relaxation modulus as

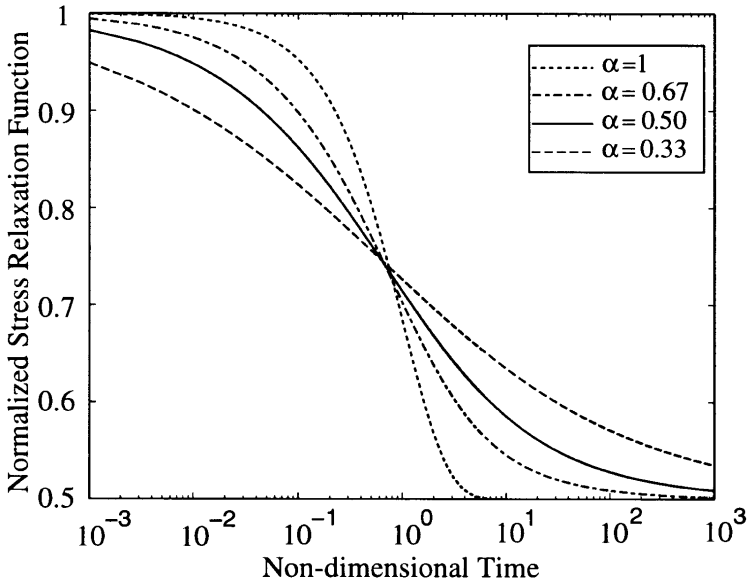

Fig. 2. Normalized stress relaxation function $G(t)/E$ vs non-dimensional time t/b. Results are given for $E/E_\infty = 2$. The influence of different values of the fractional derivative exponent α in the relaxation equation is shown.

$$G(t) = \int_0^\infty H(\tau)\, \mathrm{e}^{-t/\tau}\, \mathrm{d}(\ln \tau) + \lim_{t\to\infty} G(t) \tag{45}$$

which may be written as

$$G(t) = \int_0^\infty \frac{H(\tau)}{\tau}\, \mathrm{e}^{-t/\tau}\, \mathrm{d}\tau + E_\infty \tag{46}$$

It would be more straightforward to define the relaxation spectrum as $H(\tau)/\tau$ in eqn (46), however, this definition is not commonly used and the difference between the two definitions is only a matter of scaling. Relaxation spectra are often used for characterizing viscoelastic models derived by molecular theories. In fact, Bagley and Torvik (1983b) showed that the use of fractional derivative operators in viscoelasticity can be motivated by Rouse's molecular theory, although they did not use the concept of relaxation spectrum.

By applying a Laplace transformation in eqn (46) and using the change of variable $\tau = 1/p$, the relation between the relaxation modulus and the relaxation spectrum can be written as

$$\left[\hat{G}(s) - \frac{G_\infty}{s}\right] = \int_0^\infty \frac{H(1/p)}{p}\, \frac{1}{p+s}\, \mathrm{d}p \tag{47}$$

where $\hat{G}(s)$ is the Laplace domain relaxation modulus. By the replacement of s with $-\omega + \mathrm{i}\varepsilon$ in eqn (47) and making use of the following relation for the Dirac Delta distribution δ

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{(p-\omega)^2+\varepsilon^2}=\delta(p+\omega) \tag{48}$$

the relaxation spectrum can be expressed as

$$H(1/\omega)=-\frac{\omega}{\pi}\Im\left(\lim_{\varepsilon\to 0}\left[\hat{G}(s)-\frac{G_\infty}{s}\right]_{s=-\omega+i\varepsilon}\right) \tag{49}$$

where $\Im$ denotes imaginary part. Introducing $\hat{G}(s)$ as the Laplace transform of the relaxation modulus according to eqn (35) gives the following expression for the stress relaxation spectrum:

$$H(\tau)=\frac{\Delta E}{\pi}\frac{(\tau/b)^\alpha\sin(\pi\alpha)}{1+2(\tau/b)^\alpha\cos(\pi\alpha)+(\tau/b)^{2\alpha}},\quad 0<\alpha\pi\leqslant\pi \tag{50}$$

The peak value is given by

$$H(\tau=b)=\frac{\Delta E}{2\pi}\tan(\pi\alpha/2) \tag{51}$$

When specializing to $\alpha=1$ (i.e., a single internal variable with an integer-order relaxation function), we obtain

$$H(\tau=b)\to+\infty\quad\text{as }\alpha\to 1^-\quad\text{and}\quad\int_0^\infty\frac{H(\tau)}{\tau}\,d\tau\to\Delta E\quad\text{as }\alpha\to 1^- \tag{52}$$

and the expression for the relaxation spectrum corresponding to the single internal variable with an integer-order relaxation function may be written as

$$\frac{H(\tau)}{\tau}=\Delta E\delta(\tau-b) \tag{53}$$

which is well-known (see Nowick and Berry, 1972). This can be generalized to N 'integer-order' relaxation processes as

$$\frac{H(\tau)}{\tau}=\sum_{n=1}^{N}\Delta E_{(n)}\delta(\tau-b_{(n)}) \tag{54}$$

Figure 3 shows the relaxation spectrum according to eqn (50) for different values of α. It can be observed that the spectrum peaks more sharply as α approaches one, as expected. The relaxation constant b can now be regarded as the most probable relaxation time out of a continuous spectrum with the fractional derivative exponent α as the distribution variable, see Fig. 3. We observe that relaxation spectrum for integer-order relaxation functions are discontinuous (one peak for each process), while the fractional-order relaxation function displays a continuous relaxation spectrum. This is consistent with the idea that a continuous spectrum of damping mechanisms or internal variables might be modeled using a single mechanism or internal variable having a fractional-order evolution equation.

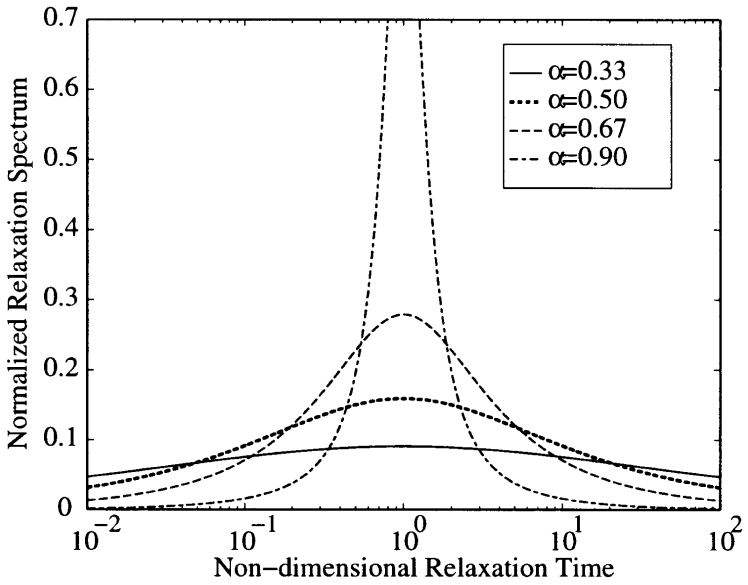

Fig. 3. Non-dimensional relaxation spectrum ($H(\tau)/\Delta E$) or distribution of non-dimensional relaxation times (τ/b). The influence of different values of the fractional derivative exponent α in the evolution equation for the anelastic strain is displayed.

4. Finite element formulation

We now consider a single anelastic displacement. For convenience we rewrite the equations governing the constitutive response in eqns (11) and (18) as

$$\boldsymbol{\sigma}(t) = K\boldsymbol{M}_{\mathrm{K}}\left(\boldsymbol{\varepsilon}(t) - \frac{\Delta K}{K}\boldsymbol{\varepsilon}^{\mathrm{A}}(t)\right) + G\boldsymbol{M}_{\mathrm{G}}\left(\boldsymbol{\varepsilon}(t) - \frac{\Delta G}{G}\boldsymbol{\varepsilon}^{\mathrm{A}}(t)\right) \tag{55}$$

$$b^{\alpha}\mathrm{D}^{\alpha}\boldsymbol{\varepsilon}^{\mathrm{A}}(t) = \boldsymbol{\varepsilon}(t) - \boldsymbol{\varepsilon}^{\mathrm{A}}(t) \quad \text{where } \boldsymbol{\varepsilon}^{\mathrm{A}} = \left(\frac{\Delta K}{3K}\boldsymbol{M}_{\mathrm{K}} + \frac{\Delta G}{2G}\boldsymbol{M}_{\mathrm{C}}\right)^{-1}\boldsymbol{\varepsilon}^{\mathrm{A}}(t) \tag{56}$$

We can express the anelastic strain quantity vector as the convolution

$$\boldsymbol{\varepsilon}^{\mathrm{A}}(t) = \int_0^t f(t-\hat{t})\boldsymbol{d}(\hat{t})\,\mathrm{d}\hat{t} \quad \text{with } f(t) = -\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{E}_{\alpha}[-(t/b)^{\alpha}]), \quad t > 0 \tag{57}$$

It is clear that the two fields above have equal spatial variation.

In accordance with a displacement based finite element formulation the total displacement field and anelastic displacement field (corresponding to $\boldsymbol{\varepsilon}^{\mathrm{A}}$) within an element are expressed in terms of interpolations of the corresponding total and anelastic degree-of-freedom vectors

$$\boldsymbol{u}^{\mathrm{e}}(\boldsymbol{x}, t) = \boldsymbol{N}^{\mathrm{e}}(\boldsymbol{x})\boldsymbol{d}^{\mathrm{e}}(t) \tag{58a}$$

$$\tilde{\boldsymbol{u}}^{\mathrm{Ae}}(\boldsymbol{x}, t) = \boldsymbol{N}^{\mathrm{e}}\tilde{\boldsymbol{d}}^{\mathrm{Ae}}(t) \tag{58b}$$

where superscript e denotes an elemental property, $\boldsymbol{N}^{\mathrm{e}}$ is the element shape function matrix, $\boldsymbol{d}^{\mathrm{e}}$ is the element total nodal degree-of-freedom vector and $\tilde{\boldsymbol{d}}^{\mathrm{Ae}}$ is the element anelastic degree-of-freedom vector. The element total strain vector $\boldsymbol{\varepsilon}^{\mathrm{e}}$ and the element anelastic strain vector $\tilde{\boldsymbol{\varepsilon}}^{\mathrm{Ae}}$ are related to corresponding element total nodal degree-of-freedom vector and anelastic nodal degree-of-freedom vector, respectively, as

$$\boldsymbol{\varepsilon}^{\mathrm{e}}(\boldsymbol{x},t) = \boldsymbol{\partial}\boldsymbol{u}^{\mathrm{e}}(\boldsymbol{x},t) = \boldsymbol{\partial}\boldsymbol{N}^{\mathrm{e}}(\boldsymbol{x})\boldsymbol{d}^{\mathrm{e}}(t) = \boldsymbol{B}^{\mathrm{e}}(\boldsymbol{x})\boldsymbol{d}^{\mathrm{e}}(t) \tag{59a}$$

$$\tilde{\boldsymbol{\varepsilon}}^{\mathrm{Ae}}(\boldsymbol{x},t) = \boldsymbol{\partial}\tilde{\boldsymbol{u}}^{\mathrm{Ae}}(\boldsymbol{x},t) = \boldsymbol{\partial}\boldsymbol{N}^{\mathrm{e}}(\boldsymbol{x})\tilde{\boldsymbol{d}}^{\mathrm{Ae}}(t) = \boldsymbol{B}^{\mathrm{e}}(\boldsymbol{x})\tilde{\boldsymbol{d}}^{\mathrm{Ae}}(t) \tag{59b}$$

where $\boldsymbol{B}^{\mathrm{e}}$ is the element strain–displacement matrix and $\boldsymbol{\partial}$ is the appropriate spatial derivative matrix operator. Using the FE-formulations of the displacements and the strains in eqns (58a)–(59b) together with the constitutive equation for the stresses in eqn (55) and the principle of virtual work, it follows that the FE-equation for dynamic equilibrium can be written as

$$\boldsymbol{M}\ddot{\boldsymbol{d}}(t) + \boldsymbol{K}\boldsymbol{d}(t) - \boldsymbol{\Delta K}\tilde{\boldsymbol{d}}^{\mathrm{A}}(t) = \boldsymbol{R}(t) \tag{60}$$

where $\boldsymbol{M}$ is the consistent mass matrix ($\rho_{(k)}$ and $V^{\mathrm{e}}_{(k)}$ are the density and volume of element k, while NEL is the number of elements)

$$\boldsymbol{M} = \sum_{k=1}^{NEL} \int_{V^{\mathrm{e}}_{(k)}} (\boldsymbol{N}^{\mathrm{e}}_{(k)})^{\mathrm{T}} \rho_{(k)} \boldsymbol{N}^{\mathrm{e}}_{(k)} \,\mathrm{d}V \tag{61}$$

$\boldsymbol{K}$ is the stiffness matrix

$$\boldsymbol{K} = \sum_{k=1}^{NEL} \int_{V^{\mathrm{e}}_{(k)}} (\boldsymbol{B}^{\mathrm{e}}_{(k)})^{\mathrm{T}} (K_{(k)}\boldsymbol{M}_{\mathrm{K}} + G_{(k)}\boldsymbol{M}_{\mathrm{G}}) \boldsymbol{B}^{\mathrm{e}}_{(k)} \,\mathrm{d}V \tag{62}$$

$\boldsymbol{\Delta K}$ is the relaxation strength matrix

$$\boldsymbol{\Delta K} = \sum_{k=1}^{NEL} \int_{V^{\mathrm{e}}_{(k)}} (\boldsymbol{B}^{\mathrm{e}}_{(k)})^{\mathrm{T}} (\Delta K_{(k)}\boldsymbol{M}_{\mathrm{K}} + \Delta G_{(k)}\boldsymbol{M}_{\mathrm{G}}) \boldsymbol{B}^{\mathrm{e}}_{(k)} \,\mathrm{d}V \tag{63}$$

and $\boldsymbol{R}$ is the external load vector, consistent with external body forces $\boldsymbol{F}$ and surface tractions $\boldsymbol{\Phi}$ (S^{e}_k is the surface area of element k).

$$\boldsymbol{R}(t) = \sum_{k=1}^{NEL} \int_{V^{\mathrm{e}}_{(k)}} (\boldsymbol{N}^{\mathrm{e}})^{\mathrm{T}} \boldsymbol{F}_{(k)}(t) \,\mathrm{d}V + \int_{S^{\mathrm{e}}_{(k)}} (\boldsymbol{N}^{\mathrm{e}})^{\mathrm{T}} \boldsymbol{\Phi}_{(k)}(t) \,\mathrm{d}S \tag{64}$$

Note that the system of FE-equations above is a system of second-order differential equations in time, and thus only requires initial conditions on the physical quantities $\boldsymbol{d}$ and $\dot{\boldsymbol{d}}$. This is the same formulation obtained if the convolution integral form of the constitutive equation is used (see Enelund and Josefson, 1997).

To solve the system of FE-equations, we need an evolution equation for the anelastic displacement vector. Restricting the formulation to materials for which eqn (56) is applicable and expressing the strains in the corresponding deformations, yields the evolution equation in terms of first-order spatial derivatives as

$$b^{\alpha}\mathrm{D}^{\alpha}\boldsymbol{\partial}\tilde{\boldsymbol{u}}^{\mathrm{A}}(\boldsymbol{x},t) + \boldsymbol{\partial}\tilde{\boldsymbol{u}}^{\mathrm{A}}(\boldsymbol{x},t) - \boldsymbol{\partial}\tilde{\boldsymbol{u}}(\boldsymbol{x},t) = \boldsymbol{0} \tag{65}$$

Equation (65) may be interpreted as a first-order (in space) differential equation. Introducing the assumed strain fields eqns (59a) and (59b) into eqn (65) yields

$$\boldsymbol{B}^{\mathrm{e}} b^{\alpha} \mathrm{D}^{\alpha} \tilde{\boldsymbol{d}}^{\mathrm{Ae}}(t) + \boldsymbol{B}^{\mathrm{e}} \tilde{\boldsymbol{d}}^{\mathrm{Ae}}(t) - \boldsymbol{B}^{\mathrm{e}} \boldsymbol{d}^{\mathrm{e}}(t) = \boldsymbol{0} \tag{66}$$

Multiplying eqn (66) from the left with the matrix transpose $(\boldsymbol{B}^{\mathrm{e}})^{\mathrm{T}}$, which can be taken as a weight function, and integrating over the element volume and summarizing contributions from all the elements yields a possible finite element formulation of eqn (65) as

$$\boldsymbol{C}(b^{\alpha} \mathrm{D}^{\alpha} \tilde{\boldsymbol{d}}^{\mathrm{A}}(t) + \tilde{\boldsymbol{d}}^{\mathrm{A}}(t) - \boldsymbol{d}(t)) = \boldsymbol{0} \tag{67}$$

where $\boldsymbol{C}$ is a quadratic matrix

$$\boldsymbol{C} = \sum_{k=1}^{NEL} \int_{V^{\mathrm{e}}_{(k)}} (\boldsymbol{B}^{\mathrm{e}}_{(k)})^{\mathrm{T}} \boldsymbol{B}^{\mathrm{e}}_{(k)} \, \mathrm{d}V \tag{68}$$

Equation (67) can be considered as fulfilling the spatial first-order partial differential equation eqn (66) in a least square sense. This is the same approach as used by Enelund and Josefson (1997) to obtain a spatial FE-discretization of the convolution term. Lesieutre and Bianchini (1995) suggested to take the divergence of the evolution equations for the anelastic strains to obtain a set of evolution equations for the anelastic displacements. This approach leads effectively to the same result as eqn (67). In practice, the structure will be restrained so that the matrix $\boldsymbol{C}$ will never be singular. Boundary conditions of the anelastic displacements will be discussed in Section 4.1.

4.1. Boundary conditions

In addition to the governing partial differential equations or the spatially discretized system of FE-equations, we need boundary conditions for unique displacement solutions. This means in the FE-formulation that the structural matrices $\boldsymbol{M}$, $\boldsymbol{K}$, $\Delta\boldsymbol{K}$ and $\boldsymbol{C}$ all are singular if the structure is unsupported. The total displacement field is treated as in the elastic case, and either the (total) displacement or the surface traction should be prescribed over the entire surface. The anelastic displacement field is found to be subjected to analogous boundary conditions. Because the anelastic displacement (strain) field is only driven by coupling to the total displacement (strain) field, it is essentially an internal field, and it may not be prescribed; only the anelastic stress may be specified over the surface. Using the material constitutive equations, the result is a mixed boundary condition, which relates the anelastic strain, the total strain, and the anelastic strain rate. However, the fact that the anelastic strains are driven exclusively by the total strains constrains the response of the anelastic field in an important way. In a finite element context, the local strains have the same spatial dependence (or variation) as the total strains. This means that the anelastic displacement has a spatial dependence like that of the total displacement. This implies that the same displacement boundary conditions that are applied to the total displacement field may be applied to the anelastic displacement field.

Using the present formulation we can write the connection between the fields as [which can be simplified assuming equal spatial variation see second part of eqn (69) below]

$$\tilde{\boldsymbol{u}}^{\mathrm{A}}(\boldsymbol{x},t)=\int_0^t f(t-\hat{t})\boldsymbol{u}(\boldsymbol{x},\hat{t})\,\mathrm{d}\hat{t}\quad\text{or}\quad\tilde{\boldsymbol{d}}^{\mathrm{A}}=\int_0^t f(t-\hat{t})\boldsymbol{d}(\hat{t})\,\mathrm{d} \tag{69}$$

where $f(t)$ is the same kernel function as in eqn (57). This suggests that the 'anelastic matrices' $\boldsymbol{\Delta K}$ and $\boldsymbol{C}$ are restrained in the same manner as the stiffness matrix $\boldsymbol{K}$. In the specific case of prescribed zero boundary displacements both the total displacement field and the anelastic displacement field should be taken as zero along the surface. However, this is just one way to restrain the anelastic field $\tilde{\boldsymbol{u}}^{\mathrm{A}}$. There are probably other ways to restrain $\tilde{\boldsymbol{u}}^{\mathrm{A}}$ (or $\boldsymbol{u}^{\mathrm{A}}$).

4.2. Algorithm

By use of a time discretization of the Grünwald definition for differentiation of general order, the α-order derivative of function $x(t)$ can be numerically evaluated by the following expression (see Oldham and Spanier, 1974)

$${}^{n+1}\mathrm{D}^{\alpha}x=\frac{1}{(\Delta t)^{\alpha}}\left({}^{n+1}x+\sum_{j=1}^{n}B_j(\alpha)\,{}^{n+1-j}x\right) \tag{70}$$

with

$$B_j(\alpha)=\frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)}$$

where $t=n\Delta t$. Note that the initial value ${}^{0}x=x(t=0)$ is not included in the sum. The calculations of ratios between gamma functions are simplified by the recursion formula

$$\frac{\Gamma(j-\alpha)}{\Gamma(j+1)}=\frac{(j-1-\alpha)}{j}\frac{\Gamma(j-1-\alpha)}{\Gamma(j)} \tag{71}$$

Application of the Backward Euler approximation (in time) to eqn (67) while using eqn (70), makes it possible to calculate the anelastic nodal degree-of-freedom vector (if the present nodal degree-of-freedom vector and the complete previous history of the anelastic nodal degree-of-freedom vector are known)

$${}^{n+1}\tilde{\boldsymbol{d}}^{\mathrm{A}}=\frac{(\Delta t/b)^{\alpha}}{1+(\Delta t/b)^{\alpha}}\,{}^{n+1}\boldsymbol{d}-\frac{1}{1+(\Delta t/b)^{\alpha}}\sum_{j=1}^{n}B_j(\alpha)\,{}^{n+1-j}\tilde{\boldsymbol{d}}^{\mathrm{A}} \tag{72}$$

Note that the complete history of the anelastic nodal degrees-of-freedom should be saved and used in each time step.

Several methods for time integration of $\ddot{\boldsymbol{d}}$ and $\dot{\boldsymbol{d}}$ exists. In the present study the Newmark method will be used, due to its simplicity and well-known operational ability. The nodal degrees-of-freedom and their first-order time derivatives at time $t=(n+1\Delta)t$ are approximated by (see Cook et al., 1989)

$${}^{n+1}\boldsymbol{d}={}^{n}\boldsymbol{d}+\Delta t\,{}^{n}\dot{\boldsymbol{d}}+\frac{(\Delta t)^2}{2}\left[(1-2\beta)\,{}^{n}\ddot{\boldsymbol{d}}+2\beta\,{}^{n+1}\ddot{\boldsymbol{d}}\right] \tag{73a}$$

and

$$ {}^{n+1}\dot{\boldsymbol{d}} = {}^{n}\dot{\boldsymbol{d}} + \Delta t[(1-\gamma){}^{n}\ddot{\boldsymbol{d}} + \gamma {}^{n+1}\ddot{\boldsymbol{d}}] \tag{73b} $$

with initial conditions ${}^{0}\boldsymbol{d} = \boldsymbol{d}_0$ and ${}^{0}\dot{\boldsymbol{d}} = \boldsymbol{v}_0$. The parameters β and γ are chosen to control stability and accuracy. With the Newmark expansions for $\dot{\boldsymbol{d}}$ and $\boldsymbol{d}$ and the expression for calculating the anelastic nodal displacements [eqn (67)] together with the system of FE-equations [eqn (60)] evaluated at the end of each time step

$$ \boldsymbol{M}^{n+1}\ddot{\boldsymbol{d}} + \boldsymbol{K}^{n+1}\boldsymbol{d} - \Delta\boldsymbol{K}^{n+1}\tilde{\boldsymbol{d}}^{\mathrm{A}} = {}^{n+1}\boldsymbol{R} \tag{74} $$

is it possible to calculate the four unknowns ($\ddot{\boldsymbol{d}}, \dot{\boldsymbol{d}}, \boldsymbol{d}$ and $\tilde{\boldsymbol{d}}^{\mathrm{A}}$).

A simple scheme is obtained by using the (nearly) explicit Newmark method [i.e., letting β equal zero in eqns (73a) and (73b)]. In the undamped case the explicit Newmark method is conditionally stable for $\gamma \geqslant 1/2$ with the critical time step

$$ \Delta t_{\mathrm{crit}} = \frac{\sqrt{2/\gamma}}{\omega_{\max}} \tag{75} $$

where $\omega_{\max}$ is the highest undamped eigenfrequency of the discretized system (found by letting $\Delta K = \Delta G = 0$ which implies $\tilde{\boldsymbol{d}}^{\mathrm{A}} = 0$). Taking this time step as the critical time step in the present damped case is believed to be conservative, since the stiffness of the structure decreases for increasing times. Numerical experiments by Enelund and Josefson (1997) somewhat verifies that the stability limit for the elastic case (using instantaneous material data) can be used in the case of time integration of structures involving kind of viscoelastic material. Here we use the explicit Newmark method with $\gamma = 1/2$, which implies no numerical damping. The scheme for calculating the four unknowns becomes very simple. The initial nodal acceleration solution vector ${}^{0}\ddot{\boldsymbol{d}}$ is first calculated from the dynamic FE-equation eqn (74). The updated nodal solution vector ${}^{n+1}\boldsymbol{d}$ is calculated directly from eqn (73a). The updated nodal anelastic solution vector ${}^{n+1}\tilde{\boldsymbol{d}}^{\mathrm{A}}$ is now obtained from eqn (67). Using this the updated nodal acceleration vector ${}^{n+1}\ddot{\boldsymbol{d}}$ is calculated from eqn (74). The updated nodal velocity vector ${}^{n+1}\dot{\boldsymbol{d}}$ is calculated directly from eqn (73b) and a new time step can be taken.

4.3. Alternative FE-formulation and algorithm

The structural response of viscoelastic structures governed by the present kind of fractional derivative constitutive relations may also be obtained without using the concept of anelastic displacements, as such see Enelund et al. (1996). A suitable method for obtaining structural responses is as follows. With the application of the Backward Euler rule together with the Grünwald algorithm eqn (70) to the evolution equation for the anelastic strains eqn (67) we have

$$ {}^{n+1}\boldsymbol{\varepsilon}^{\mathrm{A}} = \frac{1}{1+(\Delta t/b)^{\alpha}} \left((\Delta t/b)^{\alpha} \left(\frac{\Delta K}{K}\boldsymbol{M}_{\mathrm{K}} + \frac{\Delta G}{G}\boldsymbol{M}_{\mathrm{C}} \right) {}^{n+1}\boldsymbol{\varepsilon} - \sum_{j=1}^{n} B_j(\alpha)\, {}^{n+1-j}\boldsymbol{\varepsilon}^{\mathrm{A}} \right) \tag{76} $$

We write the system of FE-equations as

$$ \boldsymbol{M}\ddot{\boldsymbol{d}}(t) + \boldsymbol{f}(t) = \boldsymbol{0} \quad \text{with } \boldsymbol{f}(t) = \boldsymbol{f}^{\mathrm{int}}(t) - \boldsymbol{R}(t) \tag{77} $$

where $\boldsymbol{f}^{\mathrm{int}}$ is the internal force vector corresponding to stresses $\boldsymbol{\sigma}$. Using the explicit Newmark

method we obtain the updated nodal solution ${}^{n+1}\boldsymbol{d}$ from eqn (73a). The updated spatial strain distribution within an element is now obtained from the assumed strain field [cf eqn (59a)]

$$ {}^{n+1}\boldsymbol{\varepsilon}^{\mathrm{e}} = \boldsymbol{B}^{\mathrm{e}\,n+1}\boldsymbol{d}^{\mathrm{e}} \tag{78} $$

With this and eqn (76) we obtain the updated element spatial distribution of the anelastic strain ${}^{n+1}\boldsymbol{\varepsilon}^{\mathrm{Ae}}$. The element spatial distribution of the stress ${}^{n+1}\boldsymbol{\sigma}^{\mathrm{e}}$ is now given by the constitutive equation eqn (11). The updated internal force vector is now obtained from

$$ {}^{n+1}\boldsymbol{f}^{\mathrm{int}} = \sum_{k=1}^{NEL} \int_{V^{\mathrm{e}}_{(k)}} (\boldsymbol{B}^{\mathrm{e}}_{(k)})^{\mathrm{T}\,n+1}\boldsymbol{\sigma}^{\mathrm{e}}_{(k)}\,\mathrm{d}V \tag{79} $$

Using this the updated nodal acceleration solution vector ${}^{n+1}\ddot{\boldsymbol{d}}$ is obtained from eqn (77). With this the updated nodal velocity solution ${}^{n+1}\dot{\boldsymbol{d}}$ is found from eqn (73b) and a new time step can be taken. The advantage of using this formulation and algorithm is that no boundary conditions on the internal fields need to be specified (however, there are generally equations for six strains, but only three anelastic displacements).

5. Numerical examples

The transient response of a fixed-free uniform viscoelastic bar is calculated by the present time domain finite element formulation using a single anelastic displacement field. The bar is modeled by five linear bar elements. Bar geometry data are: length = 0.5 m and cross-sectional area = 0.0025 m^2. The bar material has uniform viscoelastic properties and the following viscous material data is used:

$$ E = 10\ \mathrm{MPa}, \quad \Delta E = 5\ \mathrm{MPa}, \quad b = 0.02\ \mathrm{s} \quad \text{and} \quad \text{density } \rho = 1000\ \mathrm{kg/m^3}. $$

Different values of the fractional derivative exponent α in the evolution equation for the anelastic displacement are used. Figure 4 shows the material loss factor corresponding to the present viscous material data and the stress relaxation modulus is displayed in Fig. 2.

The finite bar element is derived using linear shape functions and the element matrices become

$$ \boldsymbol{K}^{\mathrm{e}} = \begin{bmatrix} k^{\mathrm{e}} & -k^{\mathrm{e}} \\ -k^{\mathrm{e}} & k^{\mathrm{e}} \end{bmatrix}, \quad \boldsymbol{\Delta K}^{\mathrm{e}} = \begin{bmatrix} \Delta k^{\mathrm{e}} & -\Delta k^{\mathrm{e}} \\ -\Delta k^{\mathrm{e}} & \Delta k^{\mathrm{e}} \end{bmatrix} \quad \text{and} \quad \boldsymbol{M}^{\mathrm{e}} = \begin{bmatrix} 2m^{\mathrm{e}} & m^{\mathrm{e}} \\ m^{\mathrm{e}} & 2m^{\mathrm{e}} \end{bmatrix} $$

where $k^{\mathrm{e}} = EA^{\mathrm{e}}/L^{\mathrm{e}}$, $\Delta k^{\mathrm{e}} = \Delta E A^{\mathrm{e}}/L^{\mathrm{e}}$ and $m^{\mathrm{e}} = \rho A^{\mathrm{e}} L^{\mathrm{e}}/6$. The (elastic) critical time step for the spatially discretized bar is: $\Delta t_{\mathrm{crit}} = 0.6 \times 10^{-6}$ s. The lowest eigenfrequency of the discretized bar using instantaneous unrelaxed material data is $\omega_{\min} = 315$ rad/s, while if using long-time or relaxed material data we have $\omega_{\min} = 222$ rad/s. In all examples we normalize the tip displacement with the quasistatic long time displacement, i.e.,

$$ d_{\mathrm{stat}} = \frac{FL}{A(E-\Delta E)} = 40 \times 10^{-6}\ \mathrm{m} \tag{80} $$

where F is the applied static tip load.

In the first example the free end of bar is excited by a unit step load applied at time $t = 0$. The

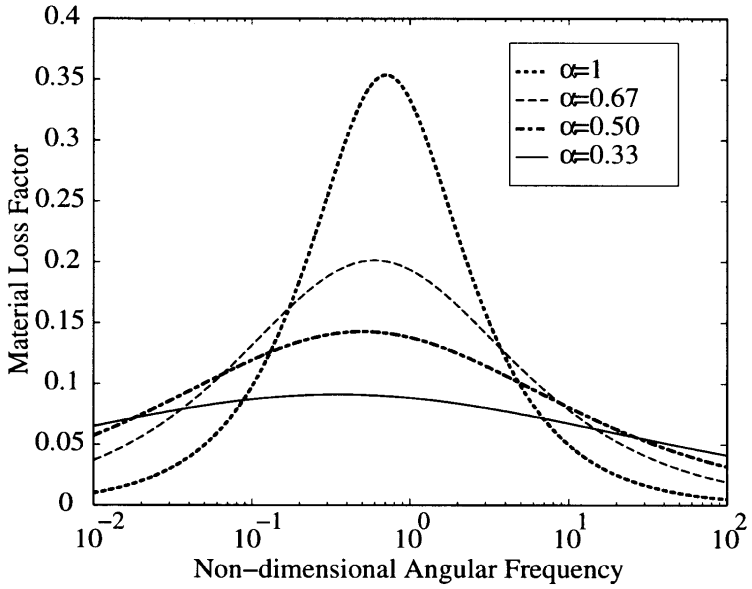

Fig. 4. Material loss factor vs non-dimensional angular frequency ωb for material data used. The influence of different values of the fractional derivative exponent α is displayed.

bar is initially at rest. The time step used is $\Delta t = 0.5\Delta t_{\text{crit}}$. Figures 5 and 6 show the tip displacement of the viscoelastic bar. The influence of different order of fractional differentiation in the evolution equation is displayed. As seen in Fig. 5 the use of a fractional-order evolution equation leads to faster decay of high frequency components. In Fig. 6 we observe that the time required to reach the quasistatic long time solution increases considerably when using a fractional-order evolution equation compared to using an integer-order evolution equation, consistent with the stress relaxation modulus in Fig. 2.

In the second example the free end of the bar is excited by a harmonic cosine load applied at time $t = 0$ with angular frequency $\Omega = 500$ rad/s. Figure 7 shows the tip displacement of the bar. As seen in the figure, high frequency components are more damped and steady state solution is reached faster with decreasing order of fractional differentiation in the evolution equation.

In all examples the fractional derivatives are obtained by the Grünwald algorithm and the complete time history of the anelastic nodal degree-of-freedom vector eqn (67) is used in each time increment. The time history can be truncated when a suitable convergence limit is reached. This can simply be done by truncating the series then the next term modifies the sum less than a small number.

6. Summary

A unique linear viscoelastic model combining the best features of anelastic displacement fields model and the fractional calculus models is presented. The evolution (relaxation) equation for the anelastic strain is taken as an in time differential equation of fractional-order. Viscoelastic function

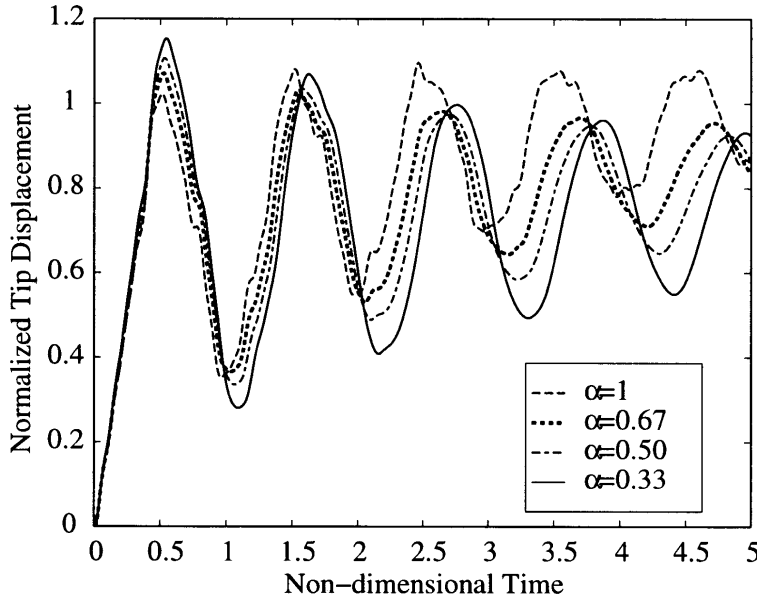

Fig. 5. Normalized tip displacement d/d_{stat} vs non-dimensional time t/b for step loaded viscoelastic bar. The load is applied at time $t = 0$. The influence of different values of the fractional derivative exponent is displayed.

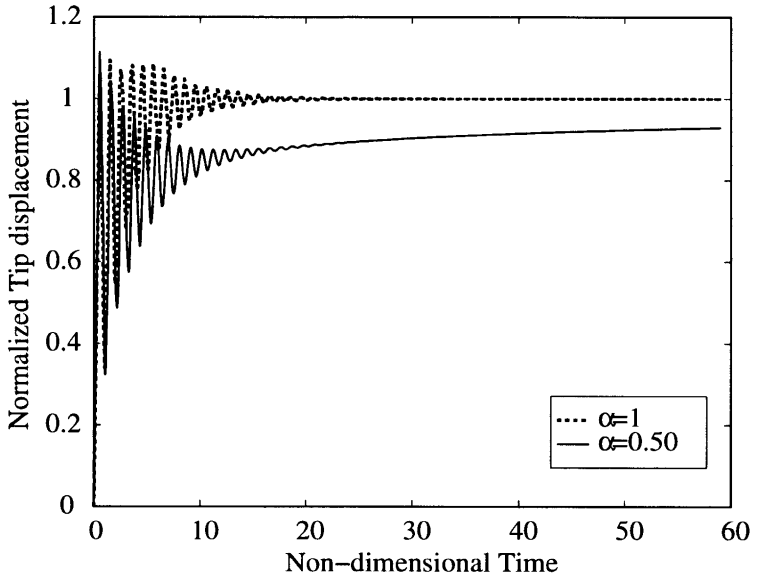

Fig. 6. Normalized tip displacement d/d_{stat} vs non-dimensional time t/b for step loaded viscoelastic bar.

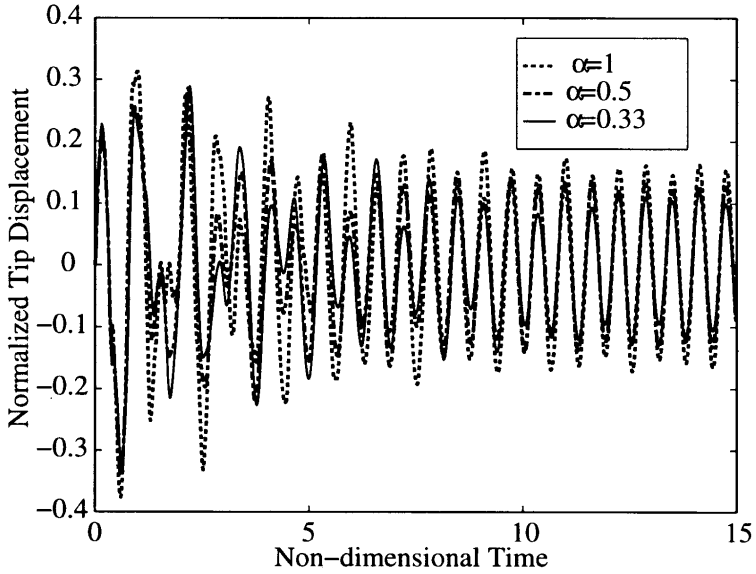

Fig. 7. Normalized tip displacement d/d_{stat} vs non-dimensional time t/b for viscoelastic bar subjected to a harmonic load applied ($\cos \Omega t$ with $\Omega = 500$ rad/s) at time $t = 0$. The influence of different values of the fractional derivative exponent α is displayed.

like dynamic modulus, stress relaxation function and relaxation spectrum are presented. From the relaxation spectrum we identify the relaxation constant in the fractional-order evolution equation for the anelastic strain as the most probable relaxation times out of a continuous distribution of relaxation times. The fractional derivative exponent then play the role of a distribution parameter. Thus, by use of a fractional-order evolution equation a whole spectrum of dissipative mechanisms may be included in a single anelastic strain. In order to illustrate the modeling capability model parameters were fitted to tabulated frequency domain data for a high-damping polymer. By using a fractional-order evolution equation the model can describe the damping characteristics of real materials over a wide frequency range using only four material parameters (in the uniaxial case).

Finite element equations governing the total displacement field and the volution of the anelastic field are outlined. The formulation requires boundary conditions on both the total and the anelastic displacement fields. The boundary conditions on the total displacement fields are the common ones in elastodynamics. The anelastic displacement field is treated as an internal field and is therefore only forced by the coupling to the total displacement field (see Lesieutre and Bianchini, 1995). The total and the anelastic fields are assumed to have equal spatial variation. With the present formulation the anelastic nodal degree-of-freedom vector is coupled to the total nodal degree-of-freedom vector through a convolution integral with a kernel of Mittag–Leffler function type. The anelastic matrices in the FE-formulation can then be restrained in the same manner as the standard stiffness matrix. A time integration algorithm for solving the FE-equations is developed. The Grünwald algorithm, for numerical fractional-order differentiation together with the Backward Euler rule are used for the time integration of the anelastic displacement field, whereby the Newmark algorithm is employed for the time integration of the structural responses.

The use of the FE-formulation and the algorithm is exemplified in some numerical examples concerning the response of a viscoelastic bar. To avoid the difficulties with boundary conditions on the anelastic field an alternative FE-formulation and algorithm for solving the FE-equations is outlined. The concept of anelastic displacement fields in not used. Instead we use the anelastic strain as the unknown internal variable. The constitutive response is then obtained by use of the Grünwald algorithm together with the Backward Euler rule, whereby the structural response is integrated by the Newmark method.

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